Hyperentangled Bell-state analysis

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It is known that it is impossible to unambiguously distinguish the four Bell states encoded in pairs of photon polarizations using linear optics. However, hyperentanglement, the simultaneous entanglement in more than one degree of freedom, has been shown to assist the complete Bell measurement of the four Bell states (given a fixed state of the other degrees of freedom). Yet introducing other degrees of freedom also enlarges the number of Bell-like states. We investigate the limits for unambiguously distinguishing a subset of all Bell-like states. In particular, we consider an additional degree of freedom to be qubit-like, such as two spatial directions, two time-bins or two orbital angular momenta, yielding 16 Bell-like states. We show that full unambiguous discrimination of these hyperentangled state is impossible. We find the optimal discrimination schemes are to group the 16 states into 7 distinguishable classes. Furthermore, we provide a procedure to uniquely distinguish any of the 16 Bell states, given two copies of it. The applications to superdense coding, quantum teleportation and fingerprinting are also discussed.

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I. INTRODUCTION

Just as Control-NOT [1] is one of the most important two-qubit gates in quantum computation, Bell-measurement is one of the most important two-qubit measurements, as it enables many applications in quantum information processing, such as superdense coding [2, 3], teleportation [4, 5, 6], quantum fingerprinting [7, 8], and direct characterization of quantum dynamics [9]. However, it was shown that complete Bell-state analysis (BSA) using linear optics is not possible [10, 11], and that the optimal probability of success is only 50% [11, 12, 13]. But Kwiat and Weinfurter (KW) [14] showed that with additional degrees of freedom, such as timing or momentum, it is indeed possible to achieve complete BSA for the four Bell states, given the additional degrees are in a fixed entangled state. other similiar BSA schemes were also proposed [15, 16, 17] and implemented [18]. In all of these schemes, such states are called hyperentangled states, and such measurements are coined embedded BSA [14]. Hyperentangled states with polarization and orbital angular momentum of two photons have recently been created and characterized [19]. Furthermore, the KW scheme for BSA has recently been implemented by Schuck et al. [20]. Nevertheless, adding additional degrees of freedom also enlarges the number of Bell-like states; all previous investigations on embedded Bell-analysis have been focused on a subset of full Bell-like states. It is, therefore, of importance to set a theoretical limit on the optimal BSA in the full set of Bell states, given fixed degrees of freedom.

In this Paper, we investigate the optimality of hyperentanglement-assisted BSA, with both degrees of freedom being qubit-like, such as polarization (H and V) plus either two momenta (spatial directions) or two orbital angular momenta. The resulting Bell-like states for two photons thus total sixteen. We show that an unambiguous state discrimination is impossible but that the optimal scheme divides the 16 Bell states into 7 distinct groups. An unambiguous discrimination of any of the sixteen states require two copies of the same states. We conclude by discussing the implications for superdense coding, teleportation and fingerprinting.

II. KWIAT-WEINFURTER SCHEME FOR BELL-STATE ANALYSIS

KW showed that when the momentum degrees of freedom are in a fixed entangled state, the four polarization Bell states can be unambiguously distinguished [14]. Here, we begin our discussion by re-analyzing their scheme, shown in Fig. 1, including all the 16 Bell-like states, constructed from two photons with the degrees of freedom being polarizations and momenta (or spatial modes): (1) $\{H,V\}\otimes\{a,c\}$ and (2) $\{H,V\}\otimes\{b,d\}$. The 16 Bell states result from the different combinations of the four polarization Bell states,

$$|\Phi^\pm\rangle \equiv \Big(|H\rangle_1|H\rangle_2 \pm |V\rangle_1|V\rangle_2\Big)/\sqrt{2}, \tag{1a}$$

$$|\Psi^\pm\rangle \equiv \Big(|H\rangle_1|V\rangle_2 \pm |V\rangle_1|H\rangle_2\Big)/\sqrt{2}, \tag{1b}$$

and the four momentum Bell states,

$$|\phi^\pm\rangle \equiv \Big(|a\rangle_1|b\rangle_2 \pm |c\rangle_1|d\rangle_2\Big)/\sqrt{2} \tag{1c}$$

$$|\psi^\pm\rangle \equiv \Big(|a\rangle_1|d\rangle_2 \pm |c\rangle_1|b\rangle_2\Big)/\sqrt{2}. \tag{1d}$$

The detection patterns and the corresponding states are shown in Table I. We clearly see that the 16 states are divided into 7 distinct classes according to the measurement outcome. Except that one class contains 4 states, all others each have 2 states. Thus, no single state can be unambiguously distinguished using this scheme. For

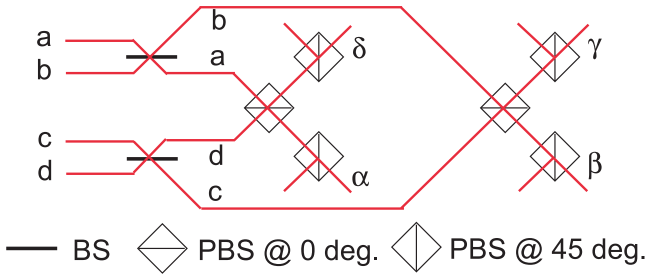

FIG. 1: (Color online) Kwiat-Weinfurter scheme for the embedded Bell-state analysis.

Class	State	Detector signature
1	$\Phi^+ \otimes \phi^+$, $\Phi^- \otimes \phi^-$ $\Psi^+ \otimes \psi^-$, $\Psi^- \otimes \psi^+$	$\alpha_{45}\alpha_{45}$, $\alpha_{\overline{45}}\alpha_{\overline{45}}$, $\beta_{45}\beta_{45}$, $\beta_{\overline{45}}\beta_{\overline{45}}$ $\delta_{45}\delta_{45}$, $\delta_{\overline{45}}\delta_{\overline{45}}$, $\gamma_{45}\gamma_{45}$, $\gamma_{\overline{45}}\gamma_{\overline{45}}$
2	$\Phi^- \otimes \phi^+$, $\Phi^+ \otimes \phi^-$	$\alpha_{45}\alpha_{\overline{45}}$, $\beta_{45}\beta_{\overline{45}}$, $\delta_{45}\delta_{\overline{45}}$, $\gamma_{45}\gamma_{\overline{45}}$
3	$\Psi^- \otimes \psi^-$, $\Psi^+ \otimes \psi^+$	$\alpha_{45}\beta_{45}$, $\alpha_{\overline{45}}\beta_{\overline{45}}$, $\delta_{45}\gamma_{45}$, $\delta_{\overline{45}}\gamma_{\overline{45}}$
4	$\Psi^+ \otimes \phi^+$, $\Phi^+ \otimes \psi^-$	$\alpha_{45}\delta_{45}$, $\alpha_{\overline{45}}\delta_{\overline{45}}$, $\beta_{45}\gamma_{45}$, $\beta_{\overline{45}}\gamma_{\overline{45}}$
5	$\Psi^+ \otimes \phi^-$, $\Phi^- \otimes \psi^-$	$\alpha_{45}\delta_{\overline{45}}$, $\alpha_{\overline{45}}\delta_{45}$, $\beta_{45}\gamma_{\overline{45}}$, $\beta_{\overline{45}}\gamma_{45}$
6	$\Psi^- \otimes \phi^+$, $\Phi^+ \otimes \psi^+$	$\alpha_{45}\gamma_{45}$, $\alpha_{\overline{45}}\gamma_{\overline{45}}$, $\beta_{45}\delta_{45}$, $\beta_{\overline{45}}\delta_{\overline{45}}$
7	$\Psi^- \otimes \phi^-$, $\Phi^- \otimes \psi^+$	$\alpha_{45}\gamma_{\overline{45}}$, $\alpha_{\overline{45}}\gamma_{45}$, $\beta_{45}\delta_{\overline{45}}$, $\beta_{\overline{45}}\delta_{45}$
*	$\Psi^{\pm} \otimes (a_1c_2 - b_1d_2)$	$\alpha_{45}\beta_{\overline{45}}$, $\alpha_{\overline{45}}\beta_{45}$, $\delta_{45}\gamma_{\overline{45}}$, $\delta_{\overline{45}}\gamma_{45}$

TABLE I: Detection signature table. $\Phi^{\pm} \equiv (H_1H_2 \pm V_1V_2)$, $\Psi^{\pm} \equiv (H_1V_2 \pm V_1H_2)$, $\phi^{\pm} \equiv (a_1b_2 \pm c_1d_2)$, and $\psi^{\pm} \equiv (a_1d_2 \pm c_1b_2)$. The subscript 45 indicates the port that comes directly through the polarizing beam splitter and $\overline{45}$ indicates the port that gets reflected.

the momentum state to be in ϕ^+, the four states with distinct polarization Bell states belong to four distinct classes, and hence can be distinguished. Similarly, for the polarization state to be in Φ^+, the states with four distinct momentum Bell states can be distinguished. Therefore, the same setup can perform BSA for either degree of freedom.

Upon further inspection of the table, we find that there is one class of four detector outcomes missing. This class, however, can be realized, e.g., by the following two states: $(H_1V_2 - V_1H_2)(a_1c_2 - b_1d_2)$ and $(H_1V_2 + V_1H_2)(a_1c_2 - b_1d_2)$, which reside outside the Hilbert space spanned by the 16 Bell states and are composed of photon 1 having spatial modes a and b and photon 2 having c and d.

III. OPTIMAL HYPERENTANGLED BELL-STATE ANALYSIS

A. Proof of optimality

One may wonder what is the optimal Bell-state analysis? Calsamiglia [13] showed that all POVM elements on two i-qudits of linear elements can have, at most, a Schmidt number of 2. As our hyperentangled Bell states have Schmidt number 4, the result of Calsamiglia means that no single state can be distinguished from any other and hence unambiguous and complete BSA for the 16 states is not possible. Thus, the optimal scheme groups the states in classes, in our case, at most 8 distinguishable classes. Notice that our analysis of the KW scheme identifies 7 classes; but is 7 the largest number of classes?

In principle, one can utilize the method of van Loock and Lütkenhaus [21] to test whether 8 classes can be discriminated. They showed that a necessary condition for the distinguishability of the states ψ_i and ψ_j ($i \neq j$) is

$$\langle \psi_i | c_s^{\dagger} c_s | \psi_j \rangle = 0, \tag{2}$$

where c_s is the annihilation operator, which is generally linearly composed of N modes (both input and auxiliary) via some unitary transformation,

$$c_s = \sum_{i=1}^{N} \nu_i c_i, \tag{3}$$

where ν_i's cannot be all zero. The rationale behind Eq. (2) is that in order for states ψ_i and ψ_j to be distinguishable, the remaining states after a single photon detection at mode s should maintain orthogonality. In addition, ancillary photons do not assist state discrimination if either input or auxiliary states have a fixed number of photons. This means that, for the purpose of checking Eq. (2), the number of modes N can be set as the number of input modes.

Therefore, for the setup shown in Fig. 1, we relabel the input modes as $|1\rangle \equiv |H\rangle \otimes |a\rangle$, $|2\rangle \equiv |H\rangle \otimes |c\rangle$, $|3\rangle \equiv |V\rangle \otimes |a\rangle$, $|4\rangle \equiv |V\rangle \otimes |c\rangle$, $|5\rangle \equiv |H\rangle \otimes |b\rangle$, $|6\rangle \equiv |H\rangle \otimes |d\rangle$, $|7\rangle \equiv |V\rangle \otimes |b\rangle$ and $|8\rangle \equiv |V\rangle \otimes |d\rangle$, where H and V denote the polarization degree of freedom and a, b, c and d denote the momentum/direction (or angular-momentum) degree of freedom. Thus, the Bell states can be written as

$$|\Psi^{(\mu)}\rangle = \sum_{i,j=1}^{8} W_{ij}^{(\mu)} c_i^{\dagger} c_j^{\dagger} |0\rangle, \tag{4}$$

where the symmetric matrices $W^{(\mu)}$ are 8×8 invertible (i.e., with nonzero determinant) and characterize the sixteen ($\mu = 1 \ldots 16$) Bell states.

In our case, if the optimal BSA groups the 16 Bell states into 8 classes, there exist sets of 8 states for which the conditions set by Eq. (2) are satisfied. On the other hand, if 7 is the optimal number of classes, no set of 8 states satisfy Eq. (2). To see whether the former or the latter is true, we have to check whether Eq. (2) can be satisfied for all possible combinations of 8 out of the 16 Bell states ($C_8^{16} = 12870$). (This number can indeed be further reduced by considering the group structure of operations that transform the 16 states onto themselves.) If we can show that it is the latter case, then we prove that 7 classes is optimal.

Class	State	Detector signature
1'	$\Phi^+ \otimes \phi^-$, $\Psi^- \otimes \phi^-$ $\Phi^+ \otimes \psi^+$, $\Psi^- \otimes \psi^+$	$\alpha_{45}\alpha_{45}$, $\alpha_{\overline{45}}\alpha_{\overline{45}}$, $\beta_{45}\beta_{45}$, $\beta_{\overline{45}}\beta_{\overline{45}}$, $\delta_{45}\delta_{45}$, $\delta_{\overline{45}}\delta_{\overline{45}}$, $\gamma_{45}\gamma_{45}$, $\gamma_{\overline{45}}\gamma_{\overline{45}}$
2'	$\Phi^- \otimes \phi^-$, $\Phi^- \otimes \psi^+$	$\alpha_{45}\alpha_{\overline{45}}$, $\beta_{45}\beta_{\overline{45}}$, $\delta_{45}\delta_{\overline{45}}$, $\gamma_{45}\gamma_{\overline{45}}$
3'	$\Psi^+ \otimes \phi^-$, $\Psi^+ \otimes \psi^+$	$\alpha_{45}\beta_{45}$, $\alpha_{\overline{45}}\beta_{\overline{45}}$, $\delta_{45}\gamma_{45}$, $\delta_{\overline{45}}\gamma_{\overline{45}}$
4'	$\Psi^+ \otimes \phi^+$, $\Phi^- \otimes \psi^-$	$\alpha_{45}\delta_{45}$, $\alpha_{\overline{45}}\delta_{\overline{45}}$, $\beta_{45}\gamma_{45}$, $\beta_{\overline{45}}\gamma_{\overline{45}}$
5'	$\Phi^+ \otimes \psi^-$, $\Psi^- \otimes \psi^-$	$\alpha_{45}\delta_{\overline{45}}$, $\alpha_{\overline{45}}\delta_{45}$, $\beta_{45}\gamma_{\overline{45}}$, $\beta_{\overline{45}}\gamma_{45}$
6'	$\Phi^+ \otimes \phi^+$, $\Psi^- \otimes \phi^+$	$\alpha_{45}\gamma_{45}$, $\alpha_{\overline{45}}\gamma_{\overline{45}}$, $\beta_{45}\delta_{45}$, $\beta_{\overline{45}}\delta_{\overline{45}}$
7'	$\Phi^- \otimes \phi^+$, $\Psi^+ \otimes \psi^-$	$\alpha_{45}\gamma_{\overline{45}}$, $\alpha_{\overline{45}}\gamma_{45}$, $\beta_{45}\delta_{\overline{45}}$, $\beta_{\overline{45}}\delta_{45}$

TABLE II: Detection signature table associated with the scheme in Fig. 2.

First, let us demonstrate an example by taking two states from class 1 and one from each of the other 6 classes:

$$\Phi^+ \otimes \phi^+ \sim |15\rangle + |26\rangle + |37\rangle + |48\rangle, \tag{5a}$$
$$\Phi^- \otimes \phi^- \sim |15\rangle - |26\rangle - |37\rangle + |48\rangle, \tag{5b}$$
$$\Phi^- \otimes \phi^+ \sim |15\rangle + |26\rangle - |37\rangle - |48\rangle, \tag{5c}$$
$$\Psi^- \otimes \psi^- \sim |18\rangle - |27\rangle - |36\rangle + |45\rangle, \tag{5d}$$
$$\Psi^+ \otimes \phi^+ \sim |17\rangle + |28\rangle + |35\rangle + |46\rangle, \tag{5e}$$
$$\Psi^+ \otimes \phi^- \sim |17\rangle - |28\rangle + |35\rangle - |46\rangle, \tag{5f}$$
$$\Psi^- \otimes \phi^+ \sim |17\rangle + |28\rangle - |35\rangle - |46\rangle, \tag{5g}$$
$$\Psi^- \otimes \phi^- \sim |17\rangle - |28\rangle - |35\rangle + |46\rangle, \tag{5h}$$

Applying Eq. (2) to these eight states, we have, after simplifying the equations,

$$|\nu_1| = |\nu_3|, |\nu_2| = |\nu_4|, |\nu_5| = |\nu_7|, |\nu_6| = |\nu_8| \tag{6a}$$
$$|\nu_1|^2 + |\nu_5|^2 = |\nu_2|^2 + |\nu_6|^2 \tag{6b}$$
$$\nu_7^*\nu_5 = \nu_2^*\nu_4 = \nu_6^*\nu_8 = \nu_3^*\nu_1 = 0. \tag{6c}$$

These lead to the only solution $\nu_i = 0$, which is a contradiction. This shows that one cannot discriminate any state from the eight states in Eqs. (5).

To see whether there exists a combination of 8 states such that any state in it might be distinguished from one another, we check all 12870 cases by programming `Mathematica` to examine the conditions derived from Eq. (2), supplemented by the normalization condition $\sum_i |\nu_i|^2 = 1$. This is achieved by, first, enumerating and simplifying equations generated from Eq. (2), as well as the normalization condition, and then by using the function `FindInstance[ ]` to find an instance of solutions. The feature of `FindInstance[ ]` is that it will always find a solution if there is one. For all the 12870 cases, `FindInstance[ ]` returns an empty set, showing no solution. Therefore, we conclude that it is impossible to distinguish among any set of 8 Bell-like hyperentangled states, and that 7 is the optimal, as is realized in the KW scheme.

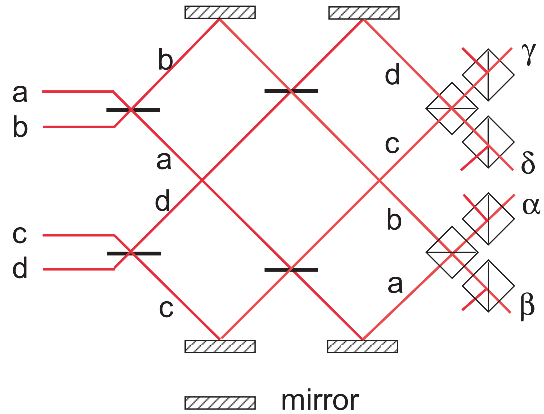

FIG. 2: (Color online) Modified Kwiat-Weinfurter scheme for Bell-state analysis.

B. Unambiguous Bell-state discrimination?

As we have seen that a one-shot measurement is unable to discriminate any Bell state, it seems natural to ask how many copies are necessary to enable the Bell-state discrimination. We show here that 2 copies are sufficient by construction. First, we introduce a slightly modified measurement scheme from KW, shown in Fig. 2. The corresponding detection patterns are shown in Table II. From Tables I and II we see that no two states share the same class of detector signature. Therefore, we let one copy go through the KW scheme and the other undergo the scheme in Fig. 2. Suppose we obtain 1 and 2'. Combining both outcomes enable us to uniquely determine which of the 16 Bell states was given, e.g., $\Phi^- \otimes \phi^-$ in the example given.

C. More degrees of freedom

We have shown that with two qubit-like degrees of freedom of two photons, there exist 7 states (out of 16) that can be distinguished from one another. Next we consider n degrees of freedom, but we shall limit ourselves to each degree of freedom being qubit-like. In this case there are 4^n Bell-like states. What is the maximum number of states that can be discriminated from one another? Again, by the results of Calsamiglia [13], a trivial upper bound is $4^n/2$. In fact, we can use the method of Calsamiglia and Lütkenhaus [12], with which they analyzed the probability of success for unambiguously discriminating any of the four-polarization Bell states. For $n \geq 2$, this probability of success is zero, as the Schmidt number of the states is 2^n, greater than 2 [13]. Even though the question we ask here is different from that of Calsamiglia and Lütkenhaus, their method still provides us a tool to seek an upper bound.

Let us begin by noting that we can express the 4^n Bell-like states in the form of Eq. (4), where the upper limit in the sum is now the number of input modes, $2n^2$. The matrices $W^{(\mu)}$ are now $(2n^2)\times(2n^2)$. If one makes a unitary transformation of the modes (using the fact that one can take the number of modes equal to the number of input modes ignoring any auxiliary mode), $a_i^\dagger = \sum_j U_{ij} c_j^\dagger$, the necessary condition for discrimination between states $\Psi^{(\mu)}$ and $\Psi^{(\nu)}$ ($\mu \neq \nu$) is

$$\langle \Psi^{(\mu)} | a_i^\dagger a_i | \Psi^{(\nu)} \rangle = 0, \tag{7}$$

or, equivalently,

$$\langle \psi_i^{(\mu)} | \psi_i^{(\nu)} \rangle = 0, \tag{8}$$

where $|\psi_i^{(\mu)}\rangle \equiv a_i |\Psi^{(\nu)}\rangle$. Because of the unitarity of W and U, $|\psi_i^{(\mu)}\rangle$ has nonzero norm and is equivalent to a $2n^2$-component vector. The above orthogonality condition then implies that there can be at most $2n^2$ linearly-independent vectors of $\psi_i^{(\mu)}$ for fixed i. Thus, we see that the maximum number of Bell states that can be distinguished is bounded above by $2n^2$. This means that the ratio of the maximal number of mutually distinguishable Bell states in a group to the total number of Bell states decreases exponentially with n: $2n^2/4^n$.

We suspect that this upper bound might be improved to $2^{n+1}-1$ (e.g., by checking all possible combinations as we did for the $n=2$ case, even though it is not practical to do for large n), as we know it is exact for $n=1$ and 2. We therefore conjecture that $N(n) \equiv 2^{n+1}-1$ is a good upper bound.

As a remark, for the case of different dimensions (instead of qubit-like) of the degrees of freedom, the upper bound can be shown to be $2d_1 d_2 \ldots d_n$.

IV. IMPLICATIONS OF HYPERENTANGLEMENT FOR QUANTUM COMMUNICATION

A. Superdense coding

Given that we can choose 7 Bell states such that they can be distinguished from one another, we can then take one of them as shared entanglement and use 7 operations, which take the state to itself or 6 others, to encode 7 messages. For example, take $\Psi^- \otimes \psi^-$ as the shared hyperentanglement between Alice and Bob. Alice can locally transform the state into 6 other states, $\Phi^+ \otimes \phi^+, \Phi^- \otimes \phi^+, \Psi^+ \otimes \phi^+, \Psi^- \otimes \phi^+, \Phi^- \otimes \psi^-$, and $\Phi^- \otimes \psi^+$. The above seven states can be distinguished using the KW scheme. Thus, Bob can perform the KW scheme to determine the encoded message by Alice, giving a superdense coding of $\log_2 7 \approx 2.8$ bits. For two photons entangled only in polarization, a superdense coding only encodes $\log_2 3 \approx 1.58$ bits [3]. Even though its extension to two pairs encodes $2\log_2 3 \approx 3.17$ bits, the four-photon detection efficiency η^4 is typically much smaller than the two-photon efficiency η^2, where η is the single-photon detection efficiency (usually much smaller than 70%). In particular, as long as the efficiency is less than $\sqrt{7/9} \approx 88\%$, the single-pair hyperentangled scheme will be superior. Thus, hyperentanglement for superdense coding seems to be of more practical use than multi-pair entanglement.

B. Quantum fingerprinting

Fingerprinting is a communication protocol in which two parties, Alice and Bob, want to test whether they receive the same message from a supplier. As they cannot have direct communication with each other, they have to communicate through a third party to test whether the two messages are the same. Instead of sending the whole messages, they send the corresponding "fingerprint" (a much shorter message) of their messages to the third party. A quantum protocol is superior to its classical counterpart because the former allows a 100% fingerprinting success. It has been shown that shared two-qubit Bell states enable perfect fingerprinting of binary-encoded $\{0,1\}$ messages [7, 8]. Here, we propose using hyperentanglement of a pair of photons to achieve perfect fingerprinting of $\{0,1,\ldots,6\}$ encoded messages.

Analogous to dense coding with hyperentanglement, Alice and Bob can share the hyperentangled state $\Psi^- \otimes \psi^-$, and both parties can locally transform the shared state into the 6 states: $\Phi^+ \otimes \phi^+, \Phi^- \otimes \phi^+, \Psi^+ \otimes \phi^+, \Psi^- \otimes \phi^+, \Phi^- \otimes \psi^-$, and $\Phi^- \otimes \psi^+$. In this way, they encode their fingerprints locally by applying the required operations. Thus, a referee can perform the Bell-analysis on the resulting two-photon state to determine whether Alice and Bob encoded the same fingerprints.

C. Quantum teleportation

A shared Bell-like state enables the teleportation of an unknown state. However, as a complete BSA of a two-photon polarization state alone is not possible, schemes employing additional degrees of freedom have been proposed [14, 15]. The embedded Bell-analysis schemes proposed in Refs. [15, 17, 18], however, cannot be used for teleportation, as their measurements do not require two photons to interfere, and can be performed locally. If their scheme could enable teleportation, it would imply that entanglement can be created locally by distant parties; but it is well known that local operations and classical communication cannot generate entanglement. We show that the KW scheme enables the teleportation of an arbitrary state encoded either in polarization or momentum (not both) with a 50% probability of success, the same probability as the two-photon polarization BSA. Suppose a photon in Alice's lab is in a state with known momentum but unknown polarization,$|\psi\rangle = \Big(\alpha|H\rangle_1 + \beta|V\rangle_1\Big) \otimes |h\rangle_1$, where $\{h, v\}$

is used to indicate its momentum degree of freedom. Alice and Bob share the Bell state $(\Phi^+ \otimes \phi^+)_{23}$ of photons 2 and 3. If Alice performs the KW Bell-analysis scheme on photons 1 and 2, there is a 50% probability (and he knows whether it succeeds) that Bob can turn his photon into the state $(\alpha|H\rangle_1 + \beta|V\rangle_1)$ by performing the corresponding local operation according to Alice's measurement outcome and post-selecting the photon from his momentum modes b or d in $\phi^+ = (a_1 b_2 + c_1 d_2)$. Similarly, an unknown momentum state $|H\rangle \otimes (\alpha|h\rangle + \beta|v\rangle)$ can be teleported. From these results, we see that the use of hyperentanglement of photons does not offer advantages for teleportation over the conventional polarization-only teleportation [5, 6], both having only 50% probability of success.

V. CONCLUDING REMARKS

We have investigated the optimal Bell-state analysis using projective measurements in linear optics for hyperentangled Bell states. In particular, we have shown that when the additional degrees of freedom are also qubit-like, the 16 Bell-like states can be, at best, divided into 7 distinct classes. Moreover, we have provided a method to uniquely unambiguously discriminate any of the 16 Bell states, given two copies of the state. Furthermore, we have provided an upper bound for the number of Bell-like states that can be distinguished from each other. We have also discussed the implications for superdense coding, fingerprinting and teleportation. The results are relevant, as the Kwiat-Weinfurter Bell analysis scheme has recently been realized experimentally [20].

Eisert has recently proposed a method for optimizing linear-optics gates [22] that provides an upper bound for the probability of success. An open question is whether his method can alternatively prove that 7 classes is optimal. Furthermore, as the number of degrees of freedom increases, the number of Bell-like states scales exponentially. It becomes increasingly impractical to check all cases in order to show the maximum number of distinguishable states. A better upper bound than the simple estimate given here is desirable. Another relevant question concerns how POVM measurements might be used to help Bell analysis in general?

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